1. Express each number as a product of its prime factors: (i) 140 (ii) 156 (iii) 3825 (iv) 5005 (v) 7429

Solution:

(i) 140

$140 = 2 \times 70 = 2 \times 2 \times 35 = 2^2 \times 5 \times 7$.

So $140 = 2^2 \times 5 \times 7$.

(ii) 156

$156 = 2 \times 78 = 2 \times 2 \times 39 = 2^2 \times 3 \times 13$.

So $156 = 2^2 \times 3 \times 13$.

(iii) 3825

$3825 \div 5 = 765; 765 \div 5 = 153; 153 = 9 \times 17 = 3^2 \times 17$.

So $3825 = 5^2 \times 3^2 \times 17$.

$3825 = 3^2 \times 5^2 \times 17$.

(iv) 5005

$5005 = 5 \times 1001$, and $1001 = 7 \times 11 \times 13$.

So $5005 = 5 \times 7 \times 11 \times 13$.

(v) 7429

Check small primes: $7429 = 17 \times 437$, and $437 = 19 \times 23$.

So $7429 = 17 \times 19 \times 23$.

2. Find the LCM and HCF of the following pairs of integers and verify that LCM $\times$ HCF $=$ product of the two numbers. (i) 26 and 91 (ii) 510 and 92 (iii) 336 and 54

First, find the prime factorization of each number. Then, to find the HCF (Highest Common Factor), multiply the smallest powers of the common prime factors. To find the LCM (Least Common Multiple), multiply the highest powers of all prime factors. Finally, verify the formula: LCM $\times$ HCF $=$ Product of the two numbers.

(i) 26 and 91 Prime factors: $26 = 2 \times 13$ and $91 = 7 \times 13$

- $\text{HCF} = 13$
- LCM $= 2 \times 7 \times 13 = 182$
- Verification: $182 \times 13 = 2366$ and $26 \times 91 = 2366$. Verified!

(ii) 510 and 92 Prime factors: $510 = 2\times3\times5\times17$ and $92 = 22\times23$

- $\text{HCF} = 2$
- LCM $= 22\times3\times5\times17\times23 = 23460$
- Verification: $23460\times2 = 46920$ and $510\times92 = 46920$. Verified!

(iii) 336 and 54 Prime factors: $336 = 24\times3\times7$ and $54 = 2\times33$

- $\text{HCF} = 2\times3 = 6$
- LCM $= 24\times33\times7 = 3024$
- Verification: $3024\times6 = 18144$ and $336\times54 = 18144$. Verified!

3. Find the LCM and HCF of the following integers by applying the prime factorisation method. (i) 12, 15 and 21 (ii) 17, 23 and 29 (iii) 8, 9 and 25

Solution:

LCM and HCF of more than two integers (by prime factorisation)

(i) 12,15 and 21
$12 = 2^2\times3$
$15 = 3\times5$
$21 = 3\times7$
HCF $=$ common prime factors with smallest powers $= 3$.
LCM $= 2^2\times3\times5\times7 = 4\times3\times5\times7 = 420$.

(ii) 17, 23 and 29
All three are distinct primes.
$\text{HCF} = 1$.
LCM $= 17\times23\times29 = 11,339$.

(iii) 8,9 and 25
$8 = 2^3, 9 = 3^2, 25 = 5^2$.
No common prime across all three $\rightarrow \text{HCF} = 1$.
LCM $= 2^3\times3^2\times5^2 = 8\times9\times25 = 1,800$.

4. Given that HCF (306, 657) = 9, find LCM (306, 657).

Solution:

Use the formula LCM $\times$ HCF $=$ Product of the two numbers to find the LCM.
Given: $\text{HCF}(306, 657) = 9$

$\text{LCM}(306, 657) \times 9 = 306 \times 657$

$\text{LCM}(306, 657) = 9306 \times 657$

$\text{LCM}(306, 657) = 34 \times 657 = 22338$

5. Check whether 6 n can end with the digit 0 for any natural number n.

Solution:
If any number ends with the digit 0 that means it should be divisible by 5.
That is, if 6^n ends with the digit 0, then the prime factorisation of 6^n would contain the prime number 5.
Prime factors of $6^n = (2 \times 3)^n = (2)^n \times (3)^n$
We can clearly observe, 5 is not present in the prime factors of 6^n. That means 6^n will not be divisible by 5.
Therefore, 6^n cannot end with the digit 0 for any natural number n.

6. Explain why 7 × 11 × 13 + 13 and 7 × 6 × 5 × 4 × 3 × 2 × 1 + 5 are composite numbers.

Solution:

A composite number is a positive integer that has at least one divisor other than 1 and itself. In simple terms, it's not a prime number.

- $7 \times 11 \times 13 + 13$ Take out 13 as a common factor: $13 \times (7 \times 11 + 1) = 13 \times (77 + 1) = 13 \times 78$. Since this number has factors 13 and 78 (in addition to 1 and itself), it is a composite number.
- $7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 + 5$ Take out 5 as a common factor: $5 \times (7 \times 6 \times 4 \times 3 \times 2 \times 1 + 1) = 5 \times (1008 + 1) = 5 \times 1009$. Since this number has factors 5 and 1009, it is a composite number.

7. There is a circular path around a sports field. Sonia takes 18 minutes to drive one round of the field, while Ravi takes 12 minutes for the same. Suppose they both start at the same point and at the same time, and go in the same direction. After how many minutes will they meet again at the starting point?

Solution:

This is a classic LCM problem. Sonia and Ravi will meet again at the starting point after a time that is the LCM (Least Common Multiple) of their individual times.
Time for Sonia = 18 minutes.
Time for Ravi = 12 minutes.
Find the LCM of 12 and 18.
Prime factors: $12 = 22 \times 3$ and $18 = 2 \times 32$ LCM $= 22 \times 32 = 4 \times 9 = 36$
They will meet again at the starting point after 36 minutes.

EXERCISE 1.2

1. Prove that $\sqrt{5}$ is irrational.

Solution:

Step 1: Assume, for the sake of contradiction, that $\sqrt{5}$ is a rational number. This means we can write it in the form $\frac{a}{b}$, where a and b are integers with no common factors other than 1 (they are coprime), and $b \neq 0$.

$$\sqrt{5} = \frac{a}{b}$$

Step 2: Square both sides of the equation to eliminate the square root.

$$5 = \frac{a^2}{b^2}$$

$$5b^2 = a^2$$

Step 3: This equation tells us that a^2 is a multiple of 5. According to a theorem in number theory, if a prime number (p) divides a^2, then it must also divide a. Therefore, a is also a multiple of 5. We can write $a = 5c$ for some integer c.

Step 4: Substitute $a = 5c$ back into the equation from Step 2.

$$5b^2 = (5c)^2$$

$$5b^2 = 25c^2$$

$$b^2 = 5c^2$$

Step 5: This new equation shows that b^2 is a multiple of 5, which means b is also a multiple of 5.

Conclusion: We have now shown that both a and b are multiples of 5. This contradicts our initial assumption that a and b are coprime. Therefore, our assumption that $\sqrt{5}$ is rational must be false. Hence, $\sqrt{5}$ is irrational.

2. Prove that $3+2\sqrt{5}$ is irrational.

Solution:
Step 1: Assume that $3+2\sqrt{5}$ is a rational number.

$3+2\sqrt{5}=\frac{a}{b}$, where a and b are integers and $b\neq 0$.

Step 2: Rearrange the equation to isolate the known irrational part, $\sqrt{5}$.

$$2\sqrt{5}=\frac{a}{b}-3$$

$$2\sqrt{5}=\frac{a-3b}{b}$$

$$\sqrt{5}=\frac{a-3b}{2b}$$

Step 3: Since a and b are integers, the expression $\frac{a-3b}{2b}$ is a rational number.

This means we have concluded that $\sqrt{5}$ is a rational number.

Conclusion: This is a contradiction because we already know that $\sqrt{5}$ is an irrational number (from Question 1). Our initial assumption must have been wrong. Therefore, $3+2\sqrt{5}$ is irrational.

3. Prove that the following are irrationals:

(i) $\frac{1}{\sqrt{2}}$ (ii) $7\sqrt{5}$ (iii) $6+\sqrt{2}$

Solution

(i) $\frac{1}{\sqrt{2}}$

Let $\frac{1}{\sqrt{2}}$ is rational.

Therefore, we can find two integers $a,b\left(b\neq 0\right)$ such that

$\frac{1}{\sqrt{2}} = \frac{a}{b}$

$\sqrt{2} = \frac{b}{a}$

$\frac{b}{a}$ is rational as a and b are integers.

Therefore, $\sqrt{2}$ is rational which contradicts to the fact that $\sqrt{2}$ is irrational.

(ii) $7\sqrt{5}$

Let $7\sqrt{5}$ is rational.

Therefore, we can find two integers $a, b(b \neq 0)$ such that

$7\sqrt{5} = \frac{a}{b}$ for some integers a and b

$\therefore \sqrt{5} = \frac{a}{7b}$

$\frac{a}{7b}$ is rational as a and b are integers.

Therefore, $\sqrt{5}$ should be rational.

This contradicts the fact that $\sqrt{5}$ is irrational. Therefore, our assumption that $7\sqrt{5}$ is rational is false. Hence, $7\sqrt{5}$ is irrational.

(iii) $6+\sqrt{2}$

Let $6+\sqrt{2}$ be rational.

Therefore, we can find two integers $a, b(b \neq 0)$ such that

$6+\sqrt{2} = \frac{a}{b}$

$\sqrt{2} = \frac{a}{b} - 6$

Since a and b are integers, $\frac{a}{b} - 6$ is also rational and hence, $\sqrt{2}$ should be rational. This contradicts the fact that $\sqrt{2}$ is irrational. Therefore, our assumption is false and hence,